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TEMPERATURE DISTRIBUTION IN A SEMI-INFINITE SOLID UNDER A FAST-MOVING ARBITRARY HEAT SOURCE*

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Abstract—An exact solution is obtained for the two-dimensional temperature field under the influence of a fast-moving arbitrarily distributed heat source. The field is assumed to be quasi-stationary, i.e. it is time independent in the moving reference attached to the source. Moreover, a high speed approximation is employed.

As an application, the result of a thermoelastoplastic yielding problem is given.

NOMENCLATURE

- A, dimensionless a, a/a_0;
- a, half-width of heat source;
- a_0, a reference width;
- C, specific heat;
- C_0, reference value of C;
- $\left.\begin{matrix} c(\,) \\ s(\,) \end{matrix}\right\}$, Fresnel integrals $\int_0^{(\cdot)} \left\{\begin{matrix} \cos(\pi t^2/2) \\ \sin(\pi t^2/2) \end{matrix}\right\} dt$;
- $f(\theta)$, dimensionless function of temperature, K/K_0;
- $g(\theta)$, dimensionless function of temperature, $\rho C/\rho_0 C_0$;
- thermal conductivity;
- K_0, reference value of K;
- n, index
- R, Péclét number, Va_0/κ;
- Q, dimensionless heat flux, q/q_0;
- Q_n, the nth component of Q;
- $q(x_1)$, heat flux distribution;
- q_0, reference value of q;
- T, temperature;
- t, time;
- V, velocity of movement of the solid;
- x_1, x_2, Cartesian coordinates relative to the heat source;
- x_1', x_2, Cartesian coordinates relative to the material;
- z, z_0, functions defined in equation (12);
- z_2, z_{02}, functions defined in equation above (12);
- Y, function defined in equation (12);
- κ, thermal diffusivity;
- λ, coefficient of friction;
- θ, dimensionless temperature, $TK/q_0 a_0$;
- θ_n, nth component of θ;
- ρ, density;
- ρ_0, reference value of ρ;
- ξ, η, dimensionless coordinates, x_1/a_0, x_2/a_0;
- Ψ, function defined in equation (12).

INTRODUCTION

ARBITRARILY distributed heat source on surfaces of solids, which conducts heat or conducts with convection and radiation on other parts of the surface, is important in analyses of interface phenomena. The medium forming the mating surface may be fluid or solid. Such interfaces are generally dynamic, i.e. the surfaces are in relative motion [1]. In these problems, the distribution

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of heat flux entering the solid is generally unknown *a priori*. However, by considering an arbitrarily distributed heat flux in the formulation of interface problems, an artifice is thereby provided for the eventual determination of its actual distribution.

In this context, because of the complexity of problems involving interfaces, it is desirable to have exact solution. Moreover, it is advantageous to have the solution in as simple a form as possible since the success of subsequent analysis may well depend on the tractable forms of the solution. An example of such a simple form is found in the relationship between thermoelastic, surface displacement and an arbitrarily distributed heat source on the surface of a semi-infinite solid [2]. In this analysis, a high-speed

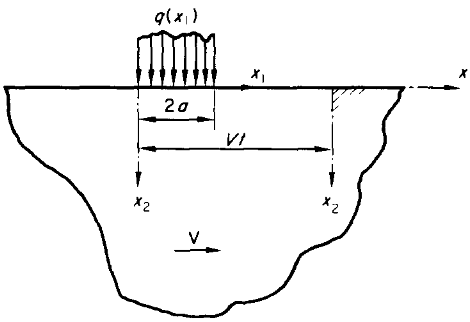

FIG. 1. Schematic diagram of the semi-infinite solid.

approximation [3] is employed to the two-dimensional heat equation under quasi-stationary conditions, i.e. time independent relative to a set of coordinates attached to the moving source.

Recent investigations on an interface phenomenon requires the knowledge of this temperature distribution. The present analysis presents the solution for the temperature distribution in a semi-infinite solid excited by a fast-moving arbitrarily distributed heat source. The source is moving on the surface of the semi-infinite solid. As an example this temperature distribution is used in the analysis of a thermo-elastoplastic yielding problem, the results of which are presented.

THE PROBLEM

Model

Figure 1 shows a semi-infinite solid moving with a constant speed, V, relative to a heat source which is arbitrarily distributed, $q(x_1)$. The coordinates (x_1', x_2) are fixed with respect to the material while (x_1, x_2) are fixed with respect to $q(x_1)$. The width of the heat source is $2a$.

Mathematical statement

The heat equation in two-dimension and for constant thermal properties is

$$\partial^2 T/\partial x_1'^2 + \partial^2 T/\partial x_2^2 = \kappa^{-1} \mathrm{D}T/\mathrm{D}t, \quad (1)$$

where T is the temperature rise over a constant reference, κ is the thermal diffusivity, t is the time and $\mathrm{D}(\ )/\mathrm{D}t$ is the material derivative. Since $x_1 = x_1' + Vt$ and $\partial T/\partial t = \mathrm{D}T/\mathrm{D}t - (\partial T/\partial x_1)\cdot(\partial x_1/\partial t)$, then for quasi-stationary state, i.e. $\partial T/\partial t = 0$, $\mathrm{D}T/\mathrm{D}t = V\partial T/\partial x_1$. For high speed V (i.e. relative to κ/a), $\partial^2 T/\partial x_1^2$ in equation (1) may be neglected [3]. Under the above restriction and since $\partial^2 T/\partial x_2'^2 = \partial^2 T/\partial x_2^2$, equation (1) and the appropriate boundary conditions may be put in the form

$$\partial^2\theta/\partial\eta^2 = R\partial\theta/\partial\xi, \quad (2)$$

$$-\partial\theta/\partial\eta = \begin{cases} Q(\xi) & (0 \leqslant \xi \leqslant 2A, \quad \eta = 0) \\ 0 & (-\infty < \xi < 0, \quad 2A < \xi < \infty, \quad \eta = 0), \end{cases} \quad (3)$$

$$\theta \to 0 \qquad [(\xi^2 + \eta^2)^{\frac{1}{2}} \to \infty], \quad (4)$$

where the following dimensionless quantities have been introduced:

$$\theta \equiv TK/q_0 a_0, \quad R \equiv Va_0/\kappa, \quad A \equiv a/a_0, \ \xi \equiv x_1/a_0, \quad \eta \equiv x_2/a_0, \quad \text{and} \quad Q \equiv q/q_0.$$

K is the thermal conductivity, a_0 and q_0 are reference quantities.

Analysis

The fundamental solution of the above system, i.e. the Green's function, is the classical LaPlace solution [4]. The system itself, therefore, possesses an integral representation of the general solution

$$\theta = \begin{cases} 0 & (\xi < 0) \\ (\pi R)^{-\frac{1}{2}} \int\limits_0^{\xi} Q(\xi')(\xi - \xi')^{-\frac{1}{2}} \exp[-R\eta^2/4(\xi - \xi')]\,\mathrm{d}\xi' & (0 \leqslant \xi \leqslant 2A) \\ (\pi R)^{-\frac{1}{2}} \int\limits_0^{2A} Q(\xi')(\xi - \xi')^{-\frac{1}{2}} \exp[-R\eta^2/4(\xi - \xi')]\,\mathrm{d}\xi' & (\xi \geqslant 2A). \end{cases} \tag{5}$$

Since $Q(\xi)$ is arbitrary, it may be represented by a Fourier series

$$Q(\xi) = \sum_{n=1}^{\infty} Q_n \sin\frac{n\pi\xi}{2A} \qquad (0 \leqslant \xi \leqslant 2A). \tag{6}$$

Letting θ_n be defined in such a way that

$$\theta = \sum_{n=1}^{\infty} \theta_n, \tag{7}$$

then θ_n may be obtained from equation (5) for the nth term of the series representation (6). It should be noted that $Q(\xi)$ is capable of representation by a cosine series as an alternative.

Equation (5) for a general term of equation (6) is

$$\theta_n = \left\{ \begin{array}{ll} 0 & (\xi < 0) \\ Q_n(\pi R)^{-\frac{1}{2}} I_1 & (0 \leqslant \xi \leqslant 2A) \\ Q_n(\pi R)^{-\frac{1}{2}} I_2 & (\xi > 2A), \end{array} \right\} \tag{8}$$

where

$$I_1 = \int\limits_0^{\xi} \sin\frac{n\pi\xi'}{2A}(\xi - \xi')^{-\frac{1}{2}} \exp[-R\eta^2/4(\xi - \xi')]\,\mathrm{d}\xi'$$

$$I_2 = \int\limits_0^{2A} \sin\frac{n\pi\xi'}{2A}(\xi - \xi')^{-\frac{1}{2}} \exp[-R^2\eta/4(\xi - \xi')]\,\mathrm{d}\xi'.$$

Now I_1 may be written as

$$I_1 = \frac{1}{2}\sin\frac{n\pi\xi}{2A}[I_{11} + I_{12}] + \frac{i}{2}\cos\frac{n\pi\xi}{2A}[I_{11} - I_{12}], \tag{9}$$

where

$$I_{11} = \int\limits_0^{\xi} \xi'^{-\frac{1}{2}} \exp\left(i\frac{n\pi\xi'}{2A} - \frac{R\eta^2}{4\xi'}\right)\mathrm{d}\xi'$$

$$I_{12} = \int\limits_0^{\xi} \xi'^{-\frac{1}{2}} \exp\left(-i\frac{n\pi\xi'}{2A} - \frac{R\eta^2}{4\xi'}\right)\mathrm{d}\xi'.$$

I_{11} may be expressed as

$$I_{11} = 2\int_0^{\sqrt{\xi}} \exp\left[-c_{11}^2\left(x^2 + \frac{a_1^2}{x^2}\right)\right]\mathrm{d}x, \tag{10}$$

where

$$\begin{cases} c_{11} = (n\pi/4A)^{\frac{1}{2}}(1 - i) \\ a_1 = (AR/4n\pi)^{\frac{1}{2}}(1 + i)\,\eta. \end{cases}$$

Differentiating equation (10) twice with respect to a_1, the following linear second order, ordinary differential equation may be obtained:

$$\frac{\mathrm{d}^2 I_{11}}{\mathrm{d}a_1^2} - 4c_{11}^4 I_1 = 4c_{11}^2 \frac{1}{\sqrt{\xi}} \exp\left[-c_{11}^2\left(\frac{a_1^2}{\xi} + \xi\right)\right].$$

This equation possesses the solution under the restriction that I_{11} remains finite and it takes on the limiting value at $a_1 = 0$,

$$I_{11} = \frac{\sqrt{\pi}}{2c_{11}} \exp(-2c_{11}^2 a_1)\left\{\mathrm{erf}(c_{11}\sqrt{\xi}) + \mathrm{erf}\left[\frac{c_{11}}{\sqrt{\xi}}(\xi - a_1)\right]\right\}.$$

Similarly,

$$I_{12} = \frac{\sqrt{\pi}}{2c_{12}} \exp(-2c_{12}^2 a_2)\left\{\mathrm{erf}(c_{12}\sqrt{\xi}) + \mathrm{erf}\left[\frac{c_{12}}{\sqrt{\xi}}(\xi - a_2)\right]\right\},$$

where

$$c_{12} = (n\pi/4A)^{\frac{1}{2}}(1 + i)$$
$$a_2 = (AR/4n\pi)^{\frac{1}{2}}(1 - i)\,\eta.$$

Note that

$$c(z) + is(z) = \frac{1 + i}{2}\,\mathrm{erf}\left[\frac{\sqrt{\pi}}{2}(1 - i)\,z\right]$$

where

$$c(\ ) = \int_0^{(\ )} \cos(\pi t^2/2)\,\mathrm{d}t$$

and

$$s(\ ) = \int_0^{(\ )} \sin(\pi t^2/2)\,\mathrm{d}t$$

are Fresnel integrals. Further, the functions $c(z)$ and $s(z)$ satisfy the following identities:

$$\begin{cases} c(-z) = -c(z) \qquad c(iz) = ic(z) \qquad c(\bar{z}) = \overline{c(z)} \\ s(-z) = -s(z) \qquad s(iz) = -is(z) \qquad s(\bar{z}) = \overline{s(z)}. \end{cases}$$

Equation (9) may now be put into the form

$$I_1 = I_1^0 + I_1^1 + I_1^2, \tag{11}$$

where

$$I_1^0 = \sqrt{\left(\frac{A}{n}\right)} e^{-Y} [\sin \Psi\, c(z_0) - \cos \Psi\, s(z_0)]$$

$$I_1^1 = \frac{1}{2}\sqrt{\left(\frac{A}{n}\right)} e^{-Y} \{\sin \Psi\, [c(\bar{z}) + c(z)] - \cos \Psi\, [s(\bar{z}) + s(z)]\}$$

$$I_1^2 = \frac{i}{2}\sqrt{\left(\frac{A}{n}\right)} e^{-Y} \{\cos \Psi\, [c(\bar{z}) - c(z)] + \sin \Psi\, [s(\bar{z}) - s(z)]\}.$$

Or

$$I_1 = \sqrt{\left(\frac{A}{n}\right)} e^{-Y} \{\sin \Psi[c(z_0) + \operatorname{Re} c(z) + \operatorname{Im} s(z)] - \cos \Psi[s(z_0) + \operatorname{Re} s(z) - \operatorname{Im} c(z)]\}.$$

where

$$Y = \left(\frac{n\pi R}{A}\right)^{\frac{1}{2}} \eta, \qquad \Psi = \frac{n\pi\xi}{2A} - Y$$

$$z_0 = \left(\frac{n\xi}{A}\right)^{\frac{1}{2}}$$

$$z = \left(\frac{n}{A\xi}\right)^{\frac{1}{2}} \left\{\left[\xi - \left(\frac{AR}{4n\pi}\right)^{\frac{1}{2}} \eta\right] + i\left(\frac{AR}{4n\pi}\right)^{\frac{1}{2}} \eta\right\}.$$

In the same manner,

$$I_2 = \sqrt{\left(\frac{A}{n}\right)} e^{-Y} \{\sin \Psi[c(z_0) + \operatorname{Re} c(z) + \operatorname{Im} s(z) - c(z_{02}) - \operatorname{Re} c(z_2) - \operatorname{Im} s(z_2)]$$
$$- \cos \Psi[s(z_0) + \operatorname{Re} s(z) - \operatorname{Im} c(z) - s(z_{02}) - \operatorname{Re} s(z_2) + \operatorname{Im} c(z_2)]\},$$

where

$$z_{02} = \left[n\left(\frac{\xi}{A} - 2\right)\right]^{\frac{1}{2}},$$

$$z_2 = \left[\frac{n}{A(\xi - 2A)}\right]^{\frac{1}{2}} \left\{\left[\xi - 2A - \left(\frac{AR}{4n\pi}\right)^{\frac{1}{2}} \eta\right] + i\left(\frac{AR}{4n\pi}\right)^{\frac{1}{2}} \eta\right\}.$$

Therefore,

$$\theta_n = \begin{cases} 0 & (\xi < 0) \\ Q_n\left(\dfrac{A}{n\pi R}\right)^{\frac{1}{2}} \mathrm{e}^{-Y}\{\sin\Psi[c(z_0) + \mathrm{Re}\, c(z) + \mathrm{Im}\, s(z)] \\ \qquad - \cos\Psi[s(z_0) + \mathrm{Re}\, s(z) - \mathrm{Im}\, c(z)]\} & (0 \leqslant \xi \leqslant 2A) \\ Q_n\left(\dfrac{A}{n\pi R}\right)^{\frac{1}{2}} \mathrm{e}^{-Y}\{\sin\Psi[c(z_0) + \mathrm{Re}\, c(z) + \mathrm{Im}\, s(z) - c(z_{02}) \\ \quad - \mathrm{Re}\, c(z_2) - \mathrm{Im}\, s(z_2)] - \cos\Psi[s(z_0) + \mathrm{Re}\, s(z) - \mathrm{Im}\, c(z) \\ \qquad - s(z_{02}) - \mathrm{Re}\, s(z_2) + \mathrm{Im}\, c(z_2)]\}, & (\xi > 2A). \end{cases} \tag{12}$$

The high-speed approximation, which is correct asymptotically with respect to R, has been shown to be accurate for $R \geqslant 5$ within the 'boundary layer' in the vicinity of the heat source.

EXAMPLE

Thermo-mechanical model

Surface roughness plays a dominant role in many surface or interface phenomena. For example, the plastic collapse of asperities on surfaces in sliding contact, which are separated by a fluid, causes the nature of the sliding process to change from the state wherein the asperities are deformed only elastically. The outward manifestation may be a relatively high frictional resistance in the elastic case and a decreasing frictional resistance subsequent to such a collapse. Hence of interest is the influence of the thermo-processes upon the collapse of these asperities.

The thermo-mechanical model is that of an ideal elastoplastic semi-infinite solid with temperature dependent yield stress. The material is taken to be isotropic and in plane strain. All other modulii are considered temperature independent. Although the speed is high based on thermal consideration, i.e. the Péclét number $R \equiv Va_0/\kappa \gg 1$, it is still negligible when compared with the speeds of propagation of mechanical disturbances, for example, the speed of propogation of the shear waves. Thus the mechanical inertia can still be neglected.

In this model, the coefficient of friction, λ, is assumed to be load and temperature independent. The body is under the mechanical loading of $p(x_1)$ and $\lambda p(x_1)$ in the region $0 \leqslant x_1 \leqslant 2a$. It is assumed that the heat flux into the body is given by [5] the expression $V\lambda p(x_1)$. An analysis of the model described above will establish the incipient plastic yielding on the surface. It should be noted that it is the physical spreading of the material under yielding which changes certain interface phenomenon. This in turn causes a decrease in the frictional coefficient. More elaborate models which take these and other contact phenomena into consideration are available but would be outside of the province of this paper. In fact for the example details of the thermo-elastoplasticity analysis will be omitted and only the results will be shown [6].

Inasmuch as the appropriate temperature field is to be used in the uncoupled thermoplasticity theory with temperature dependent yield stress, it stands to reason the relevant temperature would be such that the thermal properties are also temperature dependent. For the case where thermal conductivity $K = K_0 f(\theta)$ and the product of density and specific heat $\rho C = \rho_0 C_0 g(\theta)$, where the subscript refers to a reference state, it has been shown [7] that such temperature dependence, however strong, may be ignored so long as $(fg)^{-\frac{1}{2}} \sim 1$. With the above provisal, solutions of equation (2) under conditions (3) and (4) may be used for the intended purpose. In other words, the use of the heat equation with thermal properties which are temperature independent in a thermo- elasto-plastic analysis where the yield stress is temperature dependent may not be inconsistent.

Temperature field

A typical contour map of the temperature field is given in Fig. 2. This is for the case $R = 444$ and $q(\xi)$ is parabolically distributed in $0 \leqslant \xi \leqslant 2A$, $A = 1$.

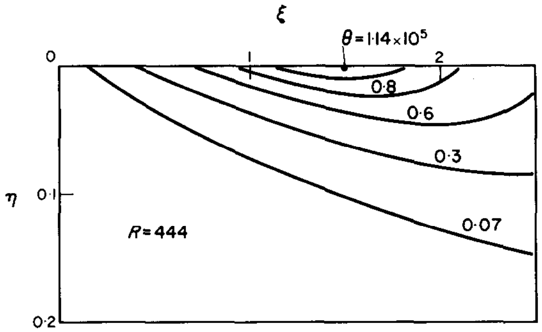

FIG. 2. Contour map of a typical temperature field in dimensionless measure $\theta(\xi, \eta)$.

Threshold of thermo-elastoplastic yielding

A thermo-elastoplastic analysis has been made in which the yield stress is a function of temperature. For this example the data for 1060 steel [8] was used. Young's modulus is 30×10^6 psi and Poisson's ratio is 0·3. Figure 3 shows the total load vs. velocity curve which is the demarcation for incipient thermo-elastoplastic yielding. This is for the case where the coefficient of friction is 0·14. The dotted line is for incipient yielding based on isothermal theory.

Note that Fig. 3 has a break near the ordinate value of 8. This is done to give some detail of the load–velocity curve while providing a measure relative to the case of isothermal yielding, i.e. elastoplatic. It is clear, within magnitudes of loads and velocities encountered in practice, that the thermal effects reported here, if neglected, could lead to drastically different conclusions.

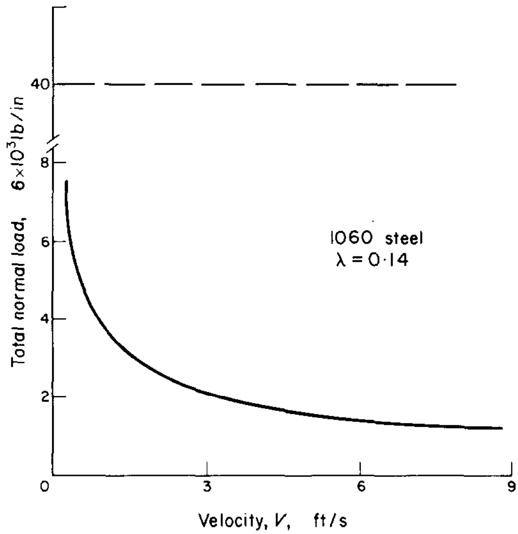

FIG. 3. Total load–velocity curve as demarcation for incipient thermoelastic plastic yielding. Coefficient of friction = 0·14. Dotted line for incipient elastoplastic yielding.

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DISTRIBUTION DE TEMPÉRATURE DANS UN SOLIDE SEMI-INFINI PAR UNE SOURCE DE CHALEUR ARBITRAIRE À DÉPLACEMENT RAPIDE

Résumé—Une solution exacte est obtenue pour le champ de température bidimensionnel sous l'influence d'une source thermique à déplacement rapide, arbitrairement distribuée. Le champ est supposé être quasi-stationnaire, c'est-à-dire indépendant du temps dans un référentiel mobile lié à la source. De plus, on utilise l'hypothèse d'une grande vitesse. En application, on considère le problème du fluage thermo-élastoplastique.

TEMPERATURVERTEILUNG IN EINEM HALBUNENDLICHEN FESTKÖRPER UNTER DEM EINFLUSS EINER SICH SCHNELL BEWEGENDEN BELIEBIGEN WÄRMEQUELLE

Zusammenfassung—Es wird eine exakte Lösung für das zweidimensionale Temperaturfeld unter dem Einfluss einer sich schnell bewegenden, willkürlich verteilten Wärmequelle gefunden. Es wird angenommen, dass das Feld quasistationär ist, d.h. es ist zeitunabhängig in einem mit der Quelle fest verbundenen Bezugssystem. Darüber hinaus wird eine rasch auswertbare Näherung verwendet.

Als Anwendungsbeispiel wird das Ergebnis einer thermoelastischen Gefügeveränderung mitgeteilt.

РАСПРЕДЕЛЕНИЕ ТЕМПЕРАТУРЫ В ПОЛУБЕСКОНЕЧНОМ ТВЕРДОМ ТЕЛЕ ПОД ВЛИЯНИЕМ БЫСТРОДВИЖУЩЕГОСЯ ПРОИЗВОЛЬНОГО ИСТОЧНИКА ТЕПЛА

Аннотация—Получено точное решение для двумерного температурного поля при наличии произвольно распределенного движущегося источника тепла. Предполагается, что поле квазистационарное, т.е.в системе отсчета, движущейся вместе с источником, оно не зависит от времени. Далее используется приближение для больших скоростей.

В качестве приложения приводятся результаты для задачи о термоупругопластической текучести.